Student Seminar Solutions Week 8

1. Show that the idele of a number field is a topological group. Show furthermore that it is a locally compact group.

Proof. Let K be a number field. For every place v of K denote by K_v the completion of K at v. For a non-archimedean place v let $\mathcal{O}_v$ be its valuation ring and $\mathcal{O}_v^\times$ its unit group. Each $K_v^\times$ is a topological group for the topology coming from the local field K_v.

The *idele group* of K is the *restricted direct product*

$$\mathbf{I}_K = \prod_v{}' K_v^\times = \left\{ (x_v)_v \in \prod_v K_v^\times \;\middle|\; x_v \in \mathcal{O}_v^\times \text{ for almost all non-archimedean } v \right\},$$

equipped with the *restricted product topology* generated by the basic open sets

$$\mathcal{U} = \prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_v^\times,$$

where S is any finite set of places containing all archimedean places, and each U_v is an open neighbourhood of a fixed element x_v in $K_v^\times$.

The goal of this note is to give a detailed proof that $\mathbf{I}_K$ is a topological group; i.e. that multiplication

$$m : \mathbf{I}_K \times \mathbf{I}_K \longrightarrow \mathbf{I}_K, \qquad m((x_v),(y_v)) = (x_v y_v)$$

and inversion

$$\iota : \mathbf{I}_K \longrightarrow \mathbf{I}_K, \qquad \iota((x_v)) = (x_v^{-1})$$

are continuous maps (and of course $\mathbf{I}_K$ is a group algebraically because multiplication and inversion are defined coordinatewise).

Key observations used repeatedly

- For each place v, the maps

  $$m_v : K_v^\times \times K_v^\times \to K_v^\times, \quad (a,b) \mapsto ab, \qquad \iota_v : K_v^\times \to K_v^\times, \quad a \mapsto a^{-1},$$

  are continuous (they are continuous in the topology of the local field K_v).

- For non-archimedean v, $\mathcal{O}_v^\times$ is a compact open subgroup of $K_v^\times$; in particular $\mathcal{O}_v^\times$ is stable under multiplication and inversion and is open in $K_v^\times$.
- A basic open set in the restricted product is obtained by choosing a finite S and open sets U_v for $v \in S$ and taking $\prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{O}_v^\times$. Equivalently such a basic open set is the intersection with the full product $\prod_v K_v^\times$ of a product-open set that equals $\mathcal{O}_v^\times$ at almost all non-archimedean v.

Continuity of multiplication:

Continuity is a local property; we prove m is continuous at an arbitrary point $\big(x, y\big) = \big((x_v)_v, (y_v)_v\big) \in \mathbf{I}_K \times \mathbf{I}_K$.

Let $\mathcal{U} \subset \mathbf{I}_K$ be a basic open neighbourhood of the product $xy = (x_v y_v)_v$. By definition there exists a finite set S of places containing all archimedean places and all non-archimedean places where either x_v or y_v fails to lie in $\mathcal{O}_v^\times$, and open sets $U_v \subset K_v^\times$ for $v \in S$, such that

$$\mathcal{U} = \prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{O}_v^\times$$

and $x_v y_v \in U_v$ for every $v \in S$.

Fix $v \in S$. Since the local multiplication m_v is continuous at (x_v, y_v) there exist open neighbourhoods V_v of x_v and W_v of y_v in $K_v^\times$ such that

$$m_v(V_v \times W_v) = \{ab : a \in V_v, b \in W_v\} \subset U_v.$$

(Explicitly: choose V_v, W_v with $V_v W_v \subset U_v$.)

For $v \notin S$ choose $V_v = W_v = \mathcal{O}_v^\times$. This choice is valid because for $v \notin S$ we have $x_v, y_v \in \mathcal{O}_v^\times$ and $\mathcal{O}_v^\times$ is a subgroup, hence $\mathcal{O}_v^\times \cdot \mathcal{O}_v^\times = \mathcal{O}_v^\times \subset \mathcal{O}_v^\times$.

Now set

$$\mathcal{V} = \prod_{v\in S} V_v \times \prod_{v\notin S} \mathcal{O}_v^\times \qquad \text{and} \qquad \mathcal{W} = \prod_{v\in S} W_v \times \prod_{v\notin S} \mathcal{O}_v^\times.$$

These are basic open neighbourhoods of x and y in $\mathbf{I}_K$, respectively. By construction, for any $(a_v)_v \in \mathcal{V}$ and $(b_v)_v \in \mathcal{W}$ we have for each place v

$$a_v b_v \in \begin{cases} U_v, & v \in S, \\ \mathcal{O}_v^\times, & v \notin S, \end{cases}$$

hence $m\big(\mathcal{V} \times \mathcal{W}\big) \subset \mathcal{U}$.

Since $\mathcal{U}$ was an arbitrary basic open neighbourhood of xy, this proves m is continuous at (x, y). Because (x, y) was arbitrary, m is continuous everywhere on $\mathbf{I}_K \times \mathbf{I}_K$.

Continuity of inversion:

Again we check continuity at an arbitrary point $x = (x_v)_v \in \mathbf{I}_K$. Let $\mathcal{U}$ be a basic open neighbourhood of $x^{-1} = (x_v^{-1})_v$. Choose a finite set S of places (containing all archimedean places and all non-archimedean places where $x_v \notin \mathcal{O}_v^\times$) and open sets $U_v \subset K_v^\times$ for $v \in S$ such that

$$\mathcal{U} = \prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_v^\times$$

and $x_v^{-1} \in U_v$ for each $v \in S$.

For each $v \in S$ the local inversion map ι_v is continuous at x_v, so there exists an open neighbourhood $V_v \subset K_v^\times$ of x_v such that $\iota_v(V_v) = \{a^{-1} : a \in V_v\} \subset U_v$. For $v \notin S$ put $V_v = \mathcal{O}_v^\times$. Note that $\mathcal{O}_v^\times$ is invariant under inversion (it is a subgroup), so $\iota_v(V_v) = \mathcal{O}_v^\times \subset \mathcal{O}_v^\times$.

Define the basic open set

$$\mathcal{V} = \prod_{v \in S} V_v \times \prod_{v \notin S} \mathcal{O}_v^\times,$$

which is an open neighbourhood of x in $\mathbf{I}_K$. By construction $\iota(\mathcal{V}) \subset \mathcal{U}$. As $\mathcal{U}$ was arbitrary, ι is continuous at x, and hence everywhere.
locally compact:
Since $\mathbf{I}_K$ is a topological group, to prove it is locally compact it suffices to find a compact neighbourhood of 1.

For each archimedean (infinite) place v, define

$$A_v = \{\, x \in K_v^\times : |x - 1|_v \leq \tfrac{1}{2} \,\},$$

which is a compact neighbourhood of 1 in $K_v^\times$.

For each non-archimedean (finite) place v, let

$$A_v = \mathcal{O}_v^\times,$$

which is compact and open in $K_v^\times$.

Now set

$$A = \prod_{v \text{ archimedean}} A_v \times \prod_{v \text{ non-archimedean}} \mathcal{O}_v^\times.$$

Let $\{U_i\}_{i \in I}$ be an open cover of A in the restricted product topology. Choose U_1 containing the identity idele $1 \in A$. Since U_1 is open in J_K, it contains a basic neighbourhood of the form

$$V = \Big(\prod_{v \in S} V_v\Big) \times \prod_{v \notin S} \mathcal{O}_v^\times,$$

where S is a finite set of places and each $V_v \subset K_v^\times$ is an open neighbourhood of 1.

Because for every non-archimedean v we have $A_v = \mathcal{O}_v^\times$, it follows that

$$\prod_{v \notin S} A_v \;=\; \prod_{v \notin S} \mathcal{O}_v^\times \;\subseteq\; V \subseteq U_1.$$

Hence U_1 covers all factors of A outside the finite set S.

Thus every element of A lies in U_1 except possibly for its projection to the finite product

$$A_S := \prod_{v \in S \cap \{\text{arch}\}} A_v \;\times \prod_{v \in S \cap \{\text{non-arch}\}} \mathcal{O}_v^\times.$$

Since S is finite and each A_v and $\mathcal{O}_v^\times$ is compact, the set A_S is a finite product of compact spaces, hence compact.

The family $\{U_i \cap A\}_{i \in I}$ covers A_S, so by compactness of A_S there exist

$$U_1,\; U_{i_2}, \ldots, U_{i_m}$$

such that

$$A_S \subset U_1 \cup U_{i_2} \cup \cdots \cup U_{i_m}.$$

Since U_1 already covers all components of A outside S, these finitely many open sets cover all of A.

Therefore $\{U_1, U_{i_2}, \ldots, U_{i_m}\}$ is a finite subcover of the original open cover, and A is compact.
Then A is a compact neighbourhood of 1 in the idele group $\mathbf{I}_K$, proving that $\mathbf{I}_K$ is locally compact. □

2. Show that $K^\times$ is a discrete subgroup of J_K. (Recall that we view $K^\times$ as a subgroup of J_K via $\iota(K^\times)$.

Proof. Let K be a number field, and as before write $\mathbf{I}_K = \prod'_v K_v^\times$ for the idele group (restricted product with respect to $\mathcal{O}_v^\times$ at finite places). The diagonal map

$$\Delta : K^\times \longrightarrow \mathbf{I}_K, \qquad \Delta(a) = (a)_v$$

is an injective group homomorphism; we identify $K^\times$ with its image $\Delta(K^\times) \subset \mathbf{I}_K$. We will show this image is a discrete subgroup of $\mathbf{I}_K$.

The subgroup $\Delta(K^\times) \subset \mathbf{I}_K$ is discrete. Equivalently, there exists a neighbourhood U of the identity $1 \in \mathbf{I}_K$ such that $U \cap \Delta(K^\times) = \{1\}$.

We construct an explicit basic neighbourhood U of 1 with the required property.

Choose for each infinite (archimedean) place v a small open neighbourhood $V_v \subset K_v$ of 0; later we will choose these so small that they separate the archimedean image of units. For each finite (non-archimedean) place v

put $W_v = \mathcal{O}_v^\times$, the unit group of the valuation ring (which is compact and open). Define a basic neighbourhood of $1 \in \mathbf{I}_K$ by

$$U = \prod_{v|\infty} (1 + V_v) \times \prod_{v\nmid\infty} \mathcal{O}_v^\times,$$

where $1 + V_v = \{1 + x : x \in V_v\} \subset K_v^\times$ (shrinking V_v if necessary we may assume $1 + V_v \subset K_v^\times$).

Suppose $a \in K^\times$ satisfies $\Delta(a) \in U$. Then for every finite place v we have $a \in \mathcal{O}_v^\times$, i.e. $v(a) = 0$. Hence a is an algebraic integer which is a unit at every finite place; consequently $a \in \mathcal{O}_K^\times$ (the global unit group of the ring of integers $\mathcal{O}_K$).

Thus

$$\Delta(K^\times) \cap U \subset \{\text{global units } \varepsilon \in \mathcal{O}_K^\times \mid \varepsilon \in 1+V_v \text{ for every archimedean } v\}.$$

Consider the archimedean embedding

$$\iota_\infty : \mathcal{O}_K^\times \longrightarrow \prod_{v|\infty} K_v^\times, \qquad \varepsilon \mapsto (\varepsilon)_{v|\infty}.$$

By Dirichlet's unit theorem (or, equivalently, by passing to the logarithmic embedding) the image of $\mathcal{O}_K^\times$ under the map

$$\varepsilon \longmapsto \big(\log |\varepsilon|_v\big)_{v|\infty}$$

is a full-rank lattice in a hyperplane of $\mathbb{R}^{r+s}$. In particular the archimedean image $\iota_\infty(\mathcal{O}_K^\times)$ is discrete in $\prod_{v|\infty} K_v^\times$ (with the product topology coming from the usual topologies on the archimedean completions).

Therefore we may choose the neighbourhoods V_v small enough so that the only global unit lying in $1 + V_v$ for every archimedean v is $\varepsilon = 1$. With such a choice we obtain

$$\Delta(K^\times) \cap U = \{1\}.$$

Hence 1 has a neighbourhood in $\mathbf{I}_K$ meeting the diagonal copy of $K^\times$ only in 1, which proves $\Delta(K^\times)$ is discrete. Since Δ is a group homomorphism, $\Delta(K^\times)$ is a discrete subgroup of $\mathbf{I}_K$.

□

3. In this exercise, you will prove the finiteness of the ideal class group $\mathscr{C}_F$. Before we do so, we introduce the content map as follows:
For an idèle $\mathbf{a} = (\ldots, a_v, \ldots) \in J_F$, define its content to be

$$\text{content}(\mathbf{a}) = \prod_{v \in V_F} ||a_v||_v$$

You may assume that the content map is a continuous homomorphism $J_F \to \mathbb{R}_+^\times$.

(a) Denote the kernel of the content map on J_F by J_F^1. Show that $F^\times < J_F^1$ and that the quotient $J_F^1/F^\times$ with the quotient topology is compact.

(b) Let $\eta : J_F \to \mathcal{I}_F$ be the map $\mathbf{a} = (\ldots, a_v, \ldots) \mapsto \prod_{v \text{ finite}} \mathfrak{p}_v^{\mathrm{ord}_v(a_v)}$. Show that if $\mathcal{I}_F$ is given the discrete topology, η is continuous.

(c) Show that $\eta(J_F^1) = \mathcal{I}_F$.

(d) Show that $\mathscr{C}_F = \mathcal{I}_F/\mathcal{P}_F$ is compact. Finish the argument by remembering that it is a discrete group.

Proof. (a) Let F be a number field. For each place v denote by F_v the completion and by $\mathcal{O}_v$ the valuation ring when v is non-archimedean. The idele group is the restricted product

$$J_F = \prod_v{}' F_v^\times,$$

and the adelic norm is

$$|\cdot| : J_F \to \mathbb{R}_{>0}, \qquad |(a_v)_v| = \prod_v |a_v|_v.$$

Put

$$J_F^1 = \ker(|\cdot|) = \{a \in J_F : \ |a| = 1\}.$$

The inclusion $F^\times \subset J_F^1$ is immediate from the product formula: for every $x \in F^\times$ one has $\prod_v |x|_v = 1$, so the diagonal image of $F^\times$ lies in J_F^1.

It remains to show compactness of $J_F^1/F^\times$. Fix some notation and a compact neighbourhood of $1 \in J_F$. For each infinite (archimedean) place v choose a compact neighbourhood $U_v \subset F_v^\times$ of 1 (for example $U_v = \{x \in F_v^\times : |x-1|_v \le 1/2\}$). For each finite (non-archimedean) place v take $U_v = \mathcal{O}_v^\times$ (compact and open). Put

$$U := \prod_{v|\infty} U_v \times \prod_{v\nmid\infty} \mathcal{O}_v^\times \subset J_F.$$

Then U is compact (Tychonoff, the first exercise) and is a neighbourhood of 1.

We will show every element of J_F^1 can be moved into U by multiplying by some element of $F^\times$. This implies

$$J_F^1 = F^\times \cdot (U \cap J_F^1),$$

so the image of the compact set $U \cap J_F^1$ in the quotient $J_F^1/F^\times$ is the whole quotient, hence $J_F^1/F^\times$ is compact.

So let $x = (x_v)_v \in J_F^1$ be arbitrary. Define the finite set S to contain all archimedean places and all finite places v for which $x_v \notin \mathcal{O}_v^\times$ (there are only finitely many such finite places by the restricted product condition). For each finite $v \in S$ choose a uniformizer $\pi_v \in F_v$ and let $n_v := \mathrm{ord}_v(x_v) \in \mathbb{Z}$. Define

$$\alpha_0 := \prod_{v \in S,\, v \nmid \infty} \pi_v^{-n_v} \in F^\times.$$

By construction the finite components of $\alpha_0 x$ are units: for every finite v,

$$\mathrm{ord}_v\big((\alpha_0 x)_v\big) = \mathrm{ord}_v(\alpha_0) + \mathrm{ord}_v(x_v) = \begin{cases} 0, & v \notin S, \\ 0, & v \in S \end{cases}$$

so $(\alpha_0 x)_v \in \mathcal{O}_v^\times$ for all finite v. Thus the only obstruction for $\alpha_0 x$ to lie in U is at the archimedean places.

We will now adjust α_0 by multiplying by a second factor $\beta \in F^\times$ which is arbitrarily close to $(\alpha_0 x_v)^{-1}$ at each archimedean v, while at the same time being congruent to 1 to sufficiently high order at each finite $v \in S$, so that the finite valuations are not altered. More precisely, choose for every infinite v an archimedean neighbourhood V_v of $(\alpha_0 x_v)^{-1}$ small enough that if $\beta \in V_v$ then $\beta(\alpha_0 x_v) \in U_v$. For each finite $v \in S$ choose an integer $N_v \geq 1$ so large that if $\beta \equiv 1 \pmod{\pi_v^{N_v}}$ then β is a unit in F_v and multiplication by β does not change the condition $(\alpha_0 x)_v \in \mathcal{O}_v^\times$.

Now apply the (standard) Weak Approximation Theorem: for the finite set of places $T := S \cup \{v \mid \infty\}$ and the prescribed targets

$$\begin{cases} \beta \equiv 1 \pmod{\pi_v^{N_v}} & \text{for each finite } v \in S, \\ \beta \in V_v & \text{for each archimedean } v, \end{cases}$$

there exists $\beta \in F^\times$ satisfying all these local conditions simultaneously. (The congruence conditions at finite places can be realized by requiring β to lie in a specified open coset in $F_v^\times$, so they fit into the local approximation conditions allowed by weak approximation.)

Set $\alpha := \beta\alpha_0 \in F^\times$. By construction:

- For every finite v, $\mathrm{ord}_v((\alpha x)_v) = \mathrm{ord}_v(\beta) + \mathrm{ord}_v(\alpha_0 x_v) = 0$, so $(\alpha x)_v \in \mathcal{O}_v^\times$.
- For every infinite v, $(\alpha x)_v = \beta(\alpha_0 x_v) \in U_v$ by the choice of V_v.

Hence $\alpha x \in U$. Since $x \in J_F^1$ was arbitrary, we have shown that every element of J_F^1 is $F^\times$-equivalent to an element of the compact set $U \cap J_F^1$.

Therefore the quotient $J_F^1/F^\times$ is compact, as required.

(b) Let $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{n_\mathfrak{p}} \in \mathcal{I}_F$ be a fixed fractional ideal. We show that $\eta^{-1}(\{\mathfrak{a}\})$ is open in J_F.

An idele $a = (a_v)_v$ satisfies $\eta(a) = \mathfrak{a}$ exactly when

$$\mathrm{ord}_\mathfrak{p}(a_\mathfrak{p}) = n_\mathfrak{p} \quad \text{for all finite primes } \mathfrak{p}.$$

Since $a_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^\times$ for all but finitely many $\mathfrak{p}$, only finitely many conditions $\mathrm{ord}_\mathfrak{p}(a_\mathfrak{p}) = n_\mathfrak{p} \neq 0$ are nontrivial.

For each finite $\mathfrak{p}$, the function

$$\mathrm{ord}_\mathfrak{p} : F_\mathfrak{p}^\times \to \mathbb{Z}$$

is locally constant: the subsets

$$U_\mathfrak{p}(m) = \{x \in F_\mathfrak{p}^\times : \mathrm{ord}_\mathfrak{p}(x) = m\} = \pi_\mathfrak{p}^m \mathcal{O}_\mathfrak{p}^\times$$

are open in $F_\mathfrak{p}^\times$.

Hence for the given $\mathfrak{a}$, the set

$$V_\mathfrak{p} = \begin{cases} \pi_\mathfrak{p}^{n_\mathfrak{p}} \mathcal{O}_\mathfrak{p}^\times, & \text{if } \mathfrak{p} \text{ is finite,} \\ F_\mathfrak{p}^\times, & \text{if } \mathfrak{p} \text{ is infinite} \end{cases}$$

is open in $F_\mathfrak{p}^\times$. Then

$$\eta^{-1}(\{\mathfrak{a}\}) = \Big(\prod_\mathfrak{p} V_\mathfrak{p}\Big) \cap J_F$$

is a basic open subset of the restricted product topology on J_F.

Since the preimage of each point $\{\mathfrak{a}\} \subset \mathcal{I}_F$ is open, η is continuous when $\mathcal{I}_F$ has the discrete topology.
(c)

Let $\mathfrak{a} = \prod_\mathfrak{p} \mathfrak{p}^{n_\mathfrak{p}} \in \mathcal{I}_F$ be any fractional ideal (only finitely many $n_\mathfrak{p} \neq 0$). We construct $a = (a_v)_v \in J_F^1$ with $\eta(a) = \mathfrak{a}$.

Step 1: prescribe finite components. Choose a finite set S of finite primes containing all $\mathfrak{p}$ with $n_\mathfrak{p} \neq 0$. For each $\mathfrak{p} \in S$ pick a uniformizer $\pi_\mathfrak{p} \in F_\mathfrak{p}$ and set

$$a_\mathfrak{p} := \pi_\mathfrak{p}^{n_\mathfrak{p}} \in F_\mathfrak{p}^\times.$$

For every finite prime $\mathfrak{p} \notin S$ set $a_\mathfrak{p} := 1 \in \mathcal{O}_\mathfrak{p}^\times$. Thus the finite part of the idele a we will build has the desired valuations:

$$\mathrm{ord}_\mathfrak{p}(a_\mathfrak{p}) = n_\mathfrak{p} \qquad (\mathfrak{p} \text{ finite}).$$

Step 2: choose archimedean components and compute the norm. For each archimedean place v choose any nonzero element $a_v \in F_v^\times$. The tuple $(a_v)_v$ (with the finite components from Step 1 and these archimedean choices) defines an idele

$$b = (b_v)_v \in J_F$$

whose finite valuations realize $\mathfrak{a}$: $\eta(b) = \mathfrak{a}$. Its adelic norm

$$t := |b| = \prod_v |b_v|_v$$

is some positive real number.

Step 3 : adjust the norm to 1 by changing an archimedean component. Let $b = (b_v)_v \in J_F$ be the idele constructed in Steps 1–2 so that $\eta(b) = \mathfrak{a}$. Set

$$t := |b| = \prod_v |b_v|_v > 0.$$

Choose an archimedean place v_0 of F (there is at least one). Since $F_{v_0} \cong \mathbb{R}$ or $\mathbb{C}$, there exists an element $c \in F_{v_0}^\times$ with $|c|_{v_0} = t^{-1}$. (Indeed, any positive real number occurs as the absolute value of some element of $\mathbb{R}^\times$ or $\mathbb{C}^\times$.)

Step 4 (conclude). Define the idele $a = (a_v)_v \in J_F$ by

$$a_v = \begin{cases} c\, b_{v_0}, & v = v_0, \\ b_v, & v \neq v_0. \end{cases}$$

Since c lies in the archimedean completion F_{v_0}, multiplying only the v_0-coordinate does not change any finite valuation, so for every finite prime $\mathfrak{p}$ we still have $\operatorname{ord}_\mathfrak{p}(a_\mathfrak{p}) = \operatorname{ord}_\mathfrak{p}(b_\mathfrak{p}) = n_\mathfrak{p}$, hence $\eta(a) = \mathfrak{a}$. Moreover

$$|a| = \left|cb_{v_0}\right|_{v_0} \prod_{v \neq v_0} |b_v|_v = |c|_{v_0} \cdot |b| = t^{-1} \cdot t = 1,$$

so $a \in J_F^1$. This proves that every fractional ideal $\mathfrak{a}$ is the content of some idele in J_F^1, i.e. $\eta(J_F^1) = \mathcal{I}_F$.

(d) From part (c) we have $\eta(J_F^1) = \mathcal{I}_F$; moreover η sends principal ideles (diagonal image of $F^\times$) to principal ideals, so η factors through the quotient by $F^\times$. Concretely, write

$$\bar{\eta} : \; J_F^1/F^\times \;\longrightarrow\; \mathcal{I}_F/\mathcal{P}_F = \mathscr{C}_F$$

for the map induced by η. The map $\bar{\eta}$ is continuous because η is continuous (when $\mathcal{I}_F$ is given the discrete topology, as shown in part (b)) and constant on $F^\times$-cosets, so it descends to the quotient.

Since $\eta(J_F^1) = \mathcal{I}_F$, the induced map $\bar{\eta}$ is surjective. A continuous image of a compact space is compact, therefore $\mathscr{C}_F = \operatorname{im}(\bar{\eta})$ is compact.

Finally, $\mathscr{C}_F$ carries the discrete topology (it is a quotient of the discrete group $\mathcal{I}_F$); a topological space that is both compact and discrete must be finite (every point is open, so compactness forces finitely many points). Hence $\mathscr{C}_F$ is finite. □

4. This exercise is to get familiar with idèles, and have some glimpses of concrete computations.
Define the map:

$$\mathbb{Q}^\times \times \mathbb{R}_+ \times \prod_p \mathbb{Z}_p^\times \to J_\mathbb{Q}$$
$$(r, t, (u_p)_p) \mapsto (rt, ru_2, ru_3, ru_5, \ldots)$$

   (a) Prove that the map is surjective.
   Hint: Given $\mathbf{a} \in J_\mathbb{Q}$, observe the element $a = \text{sign}(a_\infty) \prod_p p^{\text{ord}_p a_p}$.

   (b) Prove that the map is injective.
   Hint: It is enough to show uniqueness of the factorization of $1 = (\ldots, 1, \ldots)$.

   (c) Show that the group on the left is a topological group by exhibiting a fundamental system of neighborhoods of 1.

Proof. We consider the map

$$\Phi: \mathbb{Q}^\times \times \mathbb{R}_{>0} \times \prod_p \mathbb{Z}_p^\times \longrightarrow J_\mathbb{Q}, \qquad \Phi\big(r, t, (u_p)_p\big) = (rt,\ ru_2,\ ru_3,\ ru_5, \ldots),$$

where $J_\mathbb{Q} = \prod_v' \mathbb{Q}_v^\times$ is the idele group of $\mathbb{Q}$ (restricted product with respect to the compact open subgroups $\mathbb{Z}_p^\times$ at finite primes). We prove the three claims in the exercise.

(a) Surjectivity Let $\mathbf{a} = (a_v)_v \in J_\mathbb{Q}$. We must find $r \in \mathbb{Q}^\times$, $t \in \mathbb{R}_{>0}$ and $u_p \in \mathbb{Z}_p^\times$ for all primes p with

$$\mathbf{a} = \Phi\big(r, t, (u_p)_p\big).$$

Write the finite prime factorization of $\mathbf{a}$ at the finite places:

$$\text{ord}_p(a_p) = n_p \in \mathbb{Z}, \qquad \text{with } n_p = 0 \text{ for almost all } p.$$

Define the rational number

$$r := \text{sign}(a_\infty) \prod_p p^{n_p} \in \mathbb{Q}^\times.$$

(Here $\text{sign}(a_\infty) \in \{\pm 1\}$ is the sign of the real component a_∞.)

For each finite prime p put

$$u_p := r^{-1} a_p \in \mathbb{Q}_p^\times.$$

By construction $\operatorname{ord}_p(u_p) = \operatorname{ord}_p(a_p) - \operatorname{ord}_p(r) = n_p - n_p = 0$, hence $u_p \in \mathbb{Z}_p^\times$. At the infinite place set

$$t := r^{-1} a_\infty \in \mathbb{R}^\times.$$

Because we included $\operatorname{sign}(a_\infty)$ in the definition of r, the real number t is positive (indeed $t = |a_\infty|$), so $t \in \mathbb{R}_{>0}$.

Thus $(r, t, (u_p)_p)$ lies in the domain and

$$\Phi\big(r, t, (u_p)_p\big) = (rt,\ ru_2,\ ru_3, \ldots) = (a_\infty, a_2, a_3, \ldots) = \mathbf{a},$$

showing surjectivity.

(b) Injectivity Suppose $\Phi(r, t, (u_p))$ is the identity idele $\mathbf{1} = (1, 1, 1, \ldots)$. Then

$$rt = 1 \quad \text{in } \mathbb{R}, \qquad ru_p = 1 \quad \text{in } \mathbb{Q}_p \text{ for every } p.$$

From $ru_p = 1$ we get $u_p = r^{-1} \in \mathbb{Z}_p^\times$ for every prime p. In particular $\operatorname{ord}_p(r) = 0$ for every p, so r has no prime divisors; hence $r = \pm 1$.

From $rt = 1$ and $t > 0$ we deduce $r > 0$. Combining $r \in \{\pm 1\}$ with $r > 0$ gives $r = 1$. Therefore $t = 1$ and $u_p = 1$ for all p. This shows the only element of the domain mapping to the identity is the trivial triple, so Φ is injective.

(c) Topology: fundamental system of neighborhoods of 1 Endow the left-hand group with the product topology in which $\mathbb{Q}^\times$ is given the discrete topology, $\mathbb{R}_{>0}$ the usual topology (as a topological group under multiplication), and $\prod_p \mathbb{Z}_p^\times$ the product topology (each $\mathbb{Z}_p^\times$ being a compact open subgroup of $\mathbb{Q}_p^\times$). Each factor is a topological group, hence the product is a topological group.

A convenient fundamental system of neighbourhoods of the identity $1 = (1, 1, (1)_p)$ is given by all sets of the form

$$V_{I,\varepsilon,(N_p)} = \{1\} \times \big(1 - \varepsilon, 1 + \varepsilon\big) \times \prod_{p \in I} (1 + p^{N_p} \mathbb{Z}_p) \ \times \ \prod_{p \notin I} \mathbb{Z}_p^\times,$$

where

- I is a finite set of primes,
- each $N_p \geq 1$ is an integer,
- $\varepsilon > 0$.

Here $1 + p^{N_p}\mathbb{Z}_p$ is an open neighbourhood of 1 in $\mathbb{Z}_p^\times$, and $(1 - \varepsilon, 1 + \varepsilon)$ is an open neighbourhood of 1 in $\mathbb{R}_{>0}$ (interpreted multiplicatively). The discrete factor $\mathbb{Q}^\times$ contributes the singleton $\{1\}$ to ensure we describe neighbourhoods of the identity in the product topology.

These sets form a basis at 1 because any product neighbourhood of 1 contains one of these. Multiplication and inversion are continuous in each factor, hence continuous in the product; therefore the product is a topological group.

Finally, Φ is a bijection between the product group (with the product topology above) and $J_{\mathbb{Q}}$. One checks Φ is continuous (it is defined coordinatewise by continuous maps) and open (basic product neighbourhoods map to basic restricted-product neighbourhoods in $J_{\mathbb{Q}}$), so Φ is a topological isomorphism onto $J_{\mathbb{Q}}$. In particular the left-hand group is a topological group and identifies topologically with $J_{\mathbb{Q}}$.

□